Generalized Hermite Polynomials and the Heat Equation for Dunkl Operators

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Abstract

Based on the theory of Dunkl operators, this paper presents a general concept of multivariable Hermite polynomials and Hermite functions which are associated with finite reflection groups on $\mathbb{R}^N$. The definition and properties of these generalized Hermite systems extend naturally those of their classical counterparts; partial derivatives and the usual exponential kernel are here replaced by Dunkl operators and the generalized exponential kernel K of the Dunkl transform. In case of the symmetric group S_N, our setting includes the polynomial eigenfunctions of certain Calogero-Sutherland type operators. The second part of this paper is devoted to the heat equation associated with Dunkl's Laplacian. As in the classical case, the corresponding Cauchy problem is governed by a positive one-parameter semigroup; this is assured by a maximum principle for the generalized Laplacian. The explicit solution to the Cauchy problem involves again the kernel K, which is, on the way, proven to be nonnegative for real arguments.

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1 Introduction

Dunkl operators are differential-difference operators associated with a finite reflection group, acting on some Euclidean space. They provide a useful framework for the study of multivariable analytic structures which reveal certain reflection symmetries. During the last years, these operators have gained considerable interest in various fields of mathematics and also in physical applications; they

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are, for example, naturally connected with certain Schrödinger operators for Calogero-Sutherland-type quantum many body systems, see [L-V] and [B-F2], [B-F3].

For a finite reflection group $G \subset O(N,\mathbb{R})$ on $\mathbb{R}^N$ the associated Dunkl operators are defined as follows: For $\alpha \in \mathbb{R}^N$, denote by σ_α the reflection corresponding to α, i.e. in the hyperplane orthogonal to α. It is given by

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\,\alpha,$$

where $\langle .,.\rangle$ is the Euclidean scalar product on $\mathbb{R}^N$ and $|x| := \sqrt{\langle x,x\rangle}$. (We use the same notations for the standard Hermitian inner product and norm on $\mathbb{C}^N$.) Let R be the root system associated with the reflections of G, normalized such that $\langle \alpha,\alpha\rangle = 2$ for all $\alpha \in R$. Now choose a multiplicity function k on the root system R, that is, a G-invariant function $k: R \to \mathbb{C}$, and fix some positive subsystem R_+ of R. The Dunkl operators $T_i\,(i = 1, \ldots, N)$ on $\mathbb{R}^N$ associated with G and k are then given by

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha\in R_+} k(\alpha)\,\alpha_i \cdot \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}\,, \;\; f \in C^1(\mathbb{R}^N);$$

here ∂_i denotes the i-th partial derivative. In case $k = 0$, the T_i reduce to the corresponding partial derivatives. In this paper, we shall assume throughout that $k \geq 0$ (i.e. all values of k are non-negative), though several results of Section 3 may be extended to larger ranges of k. The most important basic properties of the T_i, proved in [D2], are as follows: Let $\mathcal{P} = \mathbb{C}\,[\mathbb{R}^N]$ denote the algebra of polynomial functions on $\mathbb{R}^N$ and $\mathcal{P}_n$ $(n \in \mathbb{Z}_+ = \{0,1,2\ldots\})$ the subspace of homogeneous polynomials of degree n. Then

(1.1) The set $\{T_i\}$ generates a commutative algebra of differential-difference operators on $\mathcal{P}$.

(1.2) Each T_i is homogeneous of degree -1 on $\mathcal{P}$, that is, $T_i\,p \in \mathcal{P}_{n-1}$ for $p \in \mathcal{P}_n$.

Of particular importance in this paper is the generalized Laplacian associated with G and k, which is defined as $\Delta_k := \sum_{i=1}^N T_i^2$. It is homogeneous of degree -2 on $\mathcal{P}$ and given explicitely by

$$\Delta_k f(x) \;=\; \Delta f(x) + 2\sum_{\alpha\in R_+} k(\alpha)\Big[\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}\Big].$$

(Here Δ and ∇ denote the usual Laplacian and gradient respectively).

The operators T_i were introduced and first studied by Dunkl in a series of papers ([Du1-4]) in connection with a generalization of the classical theory of spherical harmonics: Here the uniform spherical surface measure on the (N-1)-dimensional unit sphere is modified by a weight function which is invariant under the action of some finite reflection group G, namely

$$w_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}\,,$$

where $k \geq 0$ is some fixed multiplicity function on the root system R of G. Note that w_k is homogeneous of degree 2γ, with

$$\gamma := \sum_{\alpha \in R_+} k(\alpha).$$

In this context, in [D3] the following bilinear form on $\mathcal{P}$ is introduced:

$$[p,q]_k := (p(T)\,q)(0) \quad \text{for } p,q \in \mathcal{P}\,.$$

Here $p(T)$ is the operator derived from $p(x)$ by replacing x_i by T_i. Property (1.1) assures that $[.\,,.]_k$ is well-defined. A useful collection of its properties can be found in [D-dJ-O]. We just recall that $[.\,,.]_k$ is symmetric and positive-definite (in case $k \geq 0$), and that $[p,q]_k = 0$ for $p \in \mathcal{P}_n$, $q \in \mathcal{P}_m$ with $n \neq m$. Moreover, $[.\,,.]_k$ is closely related to the scalar product on $L^2(\mathbb{R}^N, w_k(x)e^{-|x|^2/2}dx)$: In fact, according to [D3],

$$[p,q]_k = n_k \int_{\mathbb{R}^N} e^{-\Delta_k/2}p(x)\, e^{-\Delta_k/2}q(x)\, w_k(x) e^{-|x|^2/2}dx \quad \text{for all } p,q \in \mathcal{P}, \tag{1.3}$$

with some normalization constant $n_k > 0$. Given an orthonormal basis $\{\varphi_\nu\,,\ \nu \in \mathbb{Z}_+^N\}$ of $\mathcal{P}$ with respect to $[.\,,.]_k$, an easy rescaling of (1.3) shows that the polynomials

$$H_\nu(x) := 2^{|\nu|} e^{-\Delta_k/4}\varphi_\nu$$

are orthogonal with respect to $w_k(x)e^{-|x|^2}dx$ on $\mathbb{R}^N$. We call them the generalized Hermite polynomials on $\mathbb{R}^N$ associated with G, k and $\{\varphi_\nu\}$.

The first part of this paper is devoted to the study of such Hermite polynomial systems and associated Hermite functions. They generalize their classical counterparts in a natural way: these are just obtained for $k = 0$ and $\varphi_\nu(x) = (\nu!)^{-1/2}x^\nu$. In the one-dimensional case, associated with the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$, our generalized Hermite polynomials coincide with those introduced in [Chi] and studied in [Ros]. Our setting also includes, for the symmetric group $G = S_N$, the so-called non-symmetric generalized Hermite polynomials which were recently introduced by Baker and Forrester in [B-F2], [B-F3]. These are non-symmetric analogues of Lassalle's (symmetric) generalized Hermite polynomials associated with the group S_N (see [La] and, for a further study, [B-F1]). Moreover, the "generalized Laguerre polynomials" of [B-F2], [B-F3] can be considered as a subsystem of Hermite polynomials associated with a reflection group of type B_N.

After a short collection of notations and basic facts from Dunkl's theory in Section 2, the concept of generalized Hermite polynomials is introduced in Section 3, along with a discussion of the above mentioned special classes. We derive generalizations for many of the well-known properties of the classical Hermite polynomials and Hermite functions: A Rodrigues formula, a generating relation and a Mehler formula for the Hermite polynomials, analoguesof the second order differential equations and a characterization of the generalized Hermite functions as eigenfunctions of the Dunkl transform. Parts of this section may be seen as a unifying treatment of results from [B-F2], [B-F3] and [Ros] for their particular cases.

In Section 4, which makes up the second major part of this paper, we turn to the Cauchy problem for the heat operator associated with the generalized Laplacian: Given an initial distribution $f \in C_b(\mathbb{R}^N)$, there has to be found a function $u \in C^2(\mathbb{R}^N \times (0,T)) \cap C(\mathbb{R}^N \times [0,T])$ satisfying

$$H_k u := \Delta_k u - \partial_t u = 0 \quad \text{on } \mathbb{R}^N \times (0,\infty), \qquad u(\,.\,,0) = f. \tag{1.4}$$

For smooth and rapidly decreasing initial data f an explicit solution is easy to obtain; it involves the generalized heat kernel

$$\Gamma_k(x,y,t) = \frac{M_k}{t^{\gamma+N/2}}\, e^{-(|x|^2+|y|^2)/4t}\, K\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big), \quad x,y \in \mathbb{R}^N,\ t>0.$$

Here M_k is a positive constant and K denotes the generalized exponential kernel associated with G and k as introduced in [D3]. In the theory of Dunkl operators and the Dunkl transform, it takes over the rôle of the usual exponential kernel $e^{\langle x,y\rangle}$. Some of its properties are collected in Section 2. Without knowledge whether K is nonnegative, a solution of (1.4) for arbitrary initial data seems to be difficult. However, one can prove a maximum principle for the generalized Laplacian Δ_k, which is the key ingredient to assure that Δ_k leads to a positive one-parameter contraction semigroup on the Banach space $C_0(\mathbb{R}^N), \|.\|_\infty)$. Positivity of this semigroup enforces positivity of K and allows to determine the explicit solution of (1.4) in the general case. We finish this section with an extension of a well-known maximum principle for the classical heat operator to our situation. This in particular implies a uniqueness result for solutions of the above Cauchy problem.

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2 Preliminaries

The purpose of this section is to establish our basic notations and collect some further facts on Dunkl operators and the Dunkl transform which will be of importance later on. General references here are [D3], [D4] and [dJ].

First of all we note the following product rule, which is confirmed by a short calculation: For each $f \in C^1(\mathbb{R}^N)$ and each $g \in C^1(\mathbb{R}^N)$ which is invariant under the natural action of G,

$$T_i(fg) = (T_i f)g + f(T_i g) \quad \text{for } i = 1, \ldots, N. \tag{2.1}$$

We use the common multi-index notation; in particular, for $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we set $x^\nu := x_1^{\nu_1} \cdot \ldots \cdot x_N^{\nu_N}$, $\nu! := \nu_1! \cdot \ldots \cdot \nu_N!$ and $|\nu| := \nu_1 + \ldots + \nu_N$. If $f: \mathbb{R}^N \to \mathbb{C}$ is analytic with $f(x) = \sum_\nu a_\nu x^\nu$, the operator $f(T)$ is defined by

$$f(T) := \sum_\nu a_\nu T^\nu = \sum_\nu a_\nu T_1^{\nu_1} \cdot \ldots \cdot T_N^{\nu_N}.$$

We restrict its action to $C^k(\mathbb{R}^N)$ if f is a polynomial of degree k and to $\mathcal{P}$ otherwise. The following formula will be used frequently:

2.1. Lemma. *Let $p \in \mathcal{P}_n$. Then for $c \in \mathbb{C}$ and $a \in \mathbb{C}\setminus\{0\}$,*

$$\big(e^{c\Delta_k}p\big)(ax) = a^n\big(e^{a^{-2}c\,\Delta_k}\big)p(x) \quad \textit{for all } x \in \mathbb{R}^N.$$

In particular, for $p \in \mathcal{P}_n$ we have

$$\big(e^{-\Delta_k/2}p\big)(\sqrt{2}x) \;=\; \sqrt{2}^{\,n}\big(e^{-\Delta_k/4}p\big)(x). \tag{2.2}$$

Proof. For $m \in \mathbb{Z}_+$ with $2m \le n$, the polynomial $\Delta_k^m p$ is homogeneous of degree $n-2m$. Hence

$$\big(e^{c\Delta_k}p\big)(ax) = \sum_{m=0}^{\lfloor n/2\rfloor}\frac{c^m}{m!}(\Delta_k^m p)(ax) = \sum_{m=0}^{\lfloor n/2\rfloor}\frac{c^m}{m!}a^{n-2m}(\Delta_k^m p)(x) = a^n\big(e^{a^{-2}c\,\Delta_k}p\big)(x).$$

□

[D3] a

A major tool in this paper is the generalized exponential kernel $K(x,y)$ on $\mathbb{R}^N\times\mathbb{R}^N$, which generalizes the usual exponential function $e^{\langle x,y\rangle}$. It was first introduced in [D3] by means of a certain intertwining operator. By a result of [Op1] (see also [dJ]), the function $x\mapsto K(x,y)$ may be characterized as the unique analytic solution of the system $T_if = y_if$ $(i=1,\dots,N)$ on $\mathbb{R}^N$ with $f(0)=1$. Moreover, K is symmetric in its arguments and has a holomorphic extension to $\mathbb{C}^N\times\mathbb{C}^N$. Its power series can be written as $K=\sum_{n=0}^\infty K_n$, where $K_n(x,y)=K_n(y,x)$ and K_n is a homogeneous polynomial of degree n in each of its variables. Note that $K_0=1$ and $K(z,0)=1$ for all $z\in\mathbb{C}^N$.

For the reflection group $G=\mathbb{Z}_2$ on $\mathbb{R}$, the multiplicity function k is characterized by a single parameter $\mu\ge 0$, and the kernel K is given explicitely by

$$K(z,w) = j_{\mu-1/2}(izw) - \frac{izw}{2\mu+1}\,j_{\mu+1/2}(izw), \quad z,w\in\mathbb{C},$$

where for $\alpha\ge -1/2$, j_α denotes the normalized spherical Bessel function

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} \;=\; \Gamma(\alpha+1)\cdot\sum_{n=0}^\infty\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}\,.$$

For details and related material we refer to [D4], [Rö], [Rö-V] and [Ros].

We list some further general properties of K and the K_n (all under the assumption $k\ge 0$) from [D3], [D4] and [dJ]:

For all $z,w\in\mathbb{C}^N$ and $\lambda\in\mathbb{C}$,

(2.3) $K(\lambda z,w) = K(z,\lambda w)$;

(2.4) $|K_n(z,w)| \le \dfrac{1}{n!}|z|^n|w|^n$ and $|K(z,w)| \le e^{|z||w|}$.

For all $x,y\in\mathbb{R}^N$ and $j=1,\dots,N$,

(2.5) $|K(ix,y)| \le \sqrt{|G|}$;

(2.6) $T_j^x K_n(x,y) = y_j K_{n-1}(x,y)$ and $T_j^x K(x,y) = y_j K(x,y)$;

here the superscript x denotes that the operators act with respect to the x-variable.

In [dJ], exponential bounds for the usual partial derivatives of K are given. They imply in particular that for each $\nu \in \mathbb{Z}_+^N$ there exists a constant $d_\nu > 0$, such that

(2.7) $|\partial_x^\nu K(x,z)| \,\leq\, d_\nu\, |z|^{|\nu|}\, e^{|x|\,|\mathrm{Re}\, z|}$ for all $x \in \mathbb{R}^N,\ z \in \mathbb{C}^N$.

Let us finally recall a useful reproducing kernel property of K from [D4] (it is rescaled with respect to the original one, thus fitting better in our context of generalized Hermite polynomials): Define the probability measure μ_k on $\mathbb{R}^N$ by

$$d\mu_k(x) := c_k e^{-|x|^2} w_k(x) dx, \qquad \text{with } c_k = \Big(\int_{\mathbb{R}^N} e^{-|x|^2} w_k(x) dx\Big)^{-1}.$$

Moreover, for $z \in \mathbb{C}^N$ set $l(z) := \sum_{i=1}^N z_i^2$. Then for all $z, w \in \mathbb{C}^N$,

(2.8) $\displaystyle\int_{\mathbb{R}^N} K(2z,x) K(2w,x)\, d\mu_k(x) \;=\; e^{l(z)+l(w)} K(2z,w).$

The generalized exponential function K gives rise to an integral transform, called the Dunkl transform on $\mathbb{R}^N$, which was introduced in [D4] and has been thoroughly studied in [dJ] for a large range of parameters k. The Dunkl transform associated with G and $k \geq 0$ is defined by

$$\mathcal{D}_k : L^1(\mathbb{R}^N, w_k(x)dx) \to C(\mathbb{R}^N); \;\; \mathcal{D}_k f(\xi) := \int_{\mathbb{R}^N} f(x)\, K(-i\xi, x)\, w_k(x) dx \quad (\xi \in \mathbb{R}^N).$$

In [dJ], many of the important properties of Fourier transforms on locally compact abelian groups are proved to hold true for $\mathcal{D}_k$. In particular, $\mathcal{D}_k f \in C_0(\mathbb{R}^N)$ for $f \in L^1(\mathbb{R}^N, w_k(x)dx)$, and there holds an L^1-inversion theorem, which we recall for later reference: If $f \in L^1(\mathbb{R}^N, w_k(x)dx)$ with $\mathcal{D}_k f \in L^1(\mathbb{R}^N, w_k(x)dx)$, then $f = 4^{-\gamma - N/2} c_k^2\, \mathcal{E}_k \mathcal{D}_k f$ *a.e.*, where $\mathcal{E}_k f(x) = \mathcal{D}_k f(-x)$. (Note that $\mathcal{D}_k(e^{-|x|^2/2})(0) = 2^{\gamma+N/2} c_k^{-1}$, which gives the connection of our constant c_k with that of de Jeu.) Moreover, the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$ is invariant under $\mathcal{D}_k$, and $\mathcal{D}_k$ can be extended to a Plancherel transform on $L^2(\mathbb{R}^N, w_k(x)dx)$. For details see [dJ].

3 Generalized Hermite polynomials and Hermite functions

Let $\{\varphi_\nu\,, \nu \in \mathbb{Z}_+^N\}$ be an orthonormal basis of $\mathcal{P}$ with respect to the scalar product $[\,.\,,.\,]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$ and the coefficients of the φ_ν are real. As $\mathcal{P} = \bigoplus_{n\geq 0} \mathcal{P}_n$ and $\mathcal{P}_n \perp \mathcal{P}_m$ for $n \neq m$, the φ_ν with $|\nu| = n$ can for example be constructed by Gram-Schmidt orthogonalization within $\mathcal{P}_n$ from an arbitrary ordered real-coefficient basis of $\mathcal{P}_n$. If $k = 0$, the Dunkl operator T_i reduces to the usual partial derivative ∂_i, and the canonical choice of the basis $\{\varphi_\nu\}$ is just $\varphi_\nu(x) := (\nu!)^{-1/2} x^\nu$.

As in the classical case, we have the following connection of the basis $\{\varphi_\nu\}$ with the generalized exponential function K and its homogeneous parts K_n:

3.1. Lemma. (i) $K_n(z,w) = \sum_{|\nu|=n} \varphi_\nu(z)\,\varphi_\nu(w)$ *for all* $z,w \in \mathbb{C}^N$.

(ii) $K(x,y) = \sum_{\nu\in\mathbb{Z}_+^N} \varphi_\nu(x)\,\varphi_\nu(y)$ *for all* $x,y \in \mathbb{R}^N$,

where the convergence is absolute and locally uniform on $\mathbb{R}^N \times \mathbb{R}^N$.

Proof. (i) It suffices to consider the case $z,w \in \mathbb{R}^N$. So fix some $w \in \mathbb{R}^N$. As a function of z, the polynomial $K_n(z,w)$ is homogeneous of degree n. Hence we have

$$K_n(z,w) = \sum_{|\nu|=n} c_{\nu,w}\,\varphi_\nu(z) \quad \text{with} \quad c_{\nu,w} = [K_n(\,.\,,w), \varphi_\nu]_k\,.$$

Repeated application of formula (2.6) for K_n gives

$$c_{\nu,w} = \varphi_\nu(T^z)K_n(z,w) \;=\; \varphi_\nu(w)\,K_0(z,w) \;=\; \varphi_\nu(w).$$

Thus part (i) is proved. For (ii), first note that by (2.4) we have $|K_n(x,x)| \le \frac{1}{n!}|x|^{2n}$ and hence, as the $\varphi_\nu(x)$ are real, $|\varphi_\nu(x)| \le \frac{1}{\sqrt{n!}}\,|x|^n$ for all $x \in \mathbb{R}^N$ and all ν with $|\nu| = n$. It follows that for each $M > 0$ the sum $\sum_{\mathbb{Z}_+^N} |\varphi_\nu(x)\varphi_\nu(y)|$ is majorized on $\{(x,y) : |x|,\,|y| \le M\}$ by the convergent series $\sum_{n\ge0} \binom{n+N-1}{n} M^{2n}/n!$. This yields the assertion. $\square$

For homogeneous polynomials $p,q \in \mathcal{P}_n$, relation (1.3) can be rescaled (by use of formula (2.2)):

$$[p,q]_k \;=\; 2^n \int_{\mathbb{R}^N} e^{-\Delta_k/4} p(x)\, e^{-\Delta_k/4} q(x)\, d\mu_k(x). \tag{3.1}$$

This suggests to define a generalized multivariable Hermite polynomial system on $\mathbb{R}^N$ as follows:

3.2. Definition. The generalized Hermite polynomials $\{H_\nu\,,\ \nu \in \mathbb{Z}_+^N\}$ associated with the basis $\{\varphi_\nu\}$ on $\mathbb{R}^N$ are given by

$$H_\nu(x) := 2^{|\nu|} e^{-\Delta_k/4}\varphi_\nu(x) = 2^{|\nu|} \sum_{n=0}^{\lfloor|\nu|/2\rfloor} \frac{(-1)^n}{4^n n!}\, \Delta_k^n \varphi_\nu(x). \tag{3.2}$$

Moreover, we define the generalized Hermite functions on $\mathbb{R}^N$ by

$$h_\nu(x) := e^{-|x|^2/2} H_\nu(x), \quad \nu \in \mathbb{Z}_+^N. \tag{3.3}$$

Note that H_ν is a polynomial of degree $|\nu|$ satisfying $H_\nu(-x) = (-1)^{|\nu|}H_\nu(x)$ for all $x \in \mathbb{R}^N$. A standard argument shows that $\mathcal{P}$ is dense in $L^2(\mathbb{R}^N, d\mu_k)$. Thus by virtue of (3.1), the $\{2^{-|\nu|/2}H_\nu,\ \nu \in \mathbb{Z}_+^N\}$ form an orthonormal basis of $L^2(\mathbb{R}^N, d\mu_k)$. Let us give two immediate examples:

3.3. Examples. (1) In the classical case $k = 0$ and $\varphi_\nu(x) := (\nu!)^{-1/2}x^\nu$, we obtain

$$H_\nu(x) = \frac{2^{|\nu|}}{\sqrt{\nu!}} \prod_{i=1}^N e^{-\partial_i^2/4}(x_i^{\nu_i}) \;=\; \frac{1}{\sqrt{\nu!}} \prod_{i=1}^N \widehat{H}_{\nu_i}(x_i),$$

where the $\widehat{H}_n$, $n \in \mathbb{Z}_+$ denote the classical Hermite polynomials on $\mathbb{R}$ defined by

$$e^{-x^2}\widehat{H}_n(x) = (-1)^n \frac{d^n}{dx^n}\big(e^{-x^2}\big).$$

(2) For the reflection group $G = \mathbb{Z}_2$ on $\mathbb{R}$ and multiplicity parameter $\mu \geq 0$, the polynomial basis $\{\varphi_n\}$ on $\mathbb{R}$ with respect to $[.\,,.]_\mu$ is determined uniquely (up to sign-changes) by suitable normalization of the monomials $\{x^n, n \in \mathbb{Z}_+\}$. One obtains $H_n(x) = d_n H_n^\mu(x)$, where $d_n \in \mathbb{R}\setminus\{0\}$ are constants and the H_n^μ, $n \in \mathbb{Z}_+$ are the generalized Hermite polynomials on $\mathbb{R}$ as introduced e.g. in [Chi] and studied in [Ros] (in some different normalization). They are orthogonal with respect to $|x|^{2\mu}e^{-|x|^2}$ and can be written as

$$\begin{cases} H_{2k}^\mu(x) = (-1)^k 2^{2k} k!\, L_k^{\mu-1/2}(x^2), \\ H_{2k+1}^\mu(x) = (-1)^k 2^{2k+1} k!\, x L_k^{\mu+1/2}(x^2); \end{cases}$$

here the L_n^α are the Laguerre polynomials of index $\alpha \geq -1/2$, given by

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}\Big(x^{n+\alpha}e^{-x}\Big).$$

Before discussing further examples, we are going to establish generalizations of the classical second order differential equations for Hermite polynomials and Hermite functions. For their proof we shall employ the $sl(2)$-commutation relations of the operators

$$E := \frac{1}{2}|x|^2, \;\; F := -\frac{1}{2}\Delta_k \quad \text{and} \;\; H := \sum_{i=1}^N x_i\partial_i + (\gamma + N/2)$$

on $\mathcal{P}$, which can be found e.g. in [H]; they are

$$[H, E] = 2E, \;\; [H, F] = -2F, \;\; [E, F] = H. \tag{3.4}$$

(As usual, $[A, B] = AB - BA$ for operators A, B on $\mathcal{P}$.) The first two relations are immediate consequences of the fact that the Euler operator $\rho := \sum_{i=1}^N x_i\partial_i$ satisfies $\rho(p) = np$ for each homogeneous $p \in \mathcal{P}_n$. We have the following general result:

3.4. Theorem. (1) *For $n \in \mathbb{Z}_+$ set $V_n := \{e^{-\Delta_k/4}p : p \in \mathcal{P}_n\}$. Then $\mathcal{P} = \bigoplus_{n\in\mathbb{Z}_+} V_n$, and V_n is the eigenspace of the operator $\Delta_k - 2\rho$ on $\mathcal{P}$ corresponding to the eigenvalue $-2n$.*

(2) *For $q \in V_n$, the function $f(x) := e^{-|x|^2/2}q(x)$ satisfies*

$$\big(\Delta_k - |x|^2\big) f \;=\; -(2n + 2\gamma + N) f\,.$$

Proof. (1) It is clear that $\mathcal{P} = \bigoplus V_n$. By induction from (3.4), or simply by the facts that $\rho(p) = np$ for $p \in \mathcal{P}_n$ and Δ_k is homogeneous of degree -2, we obtain the commuting relations

$$[2\rho, \Delta_k^n] = -4n\Delta_k^n \quad \text{for all } n \in \mathbb{Z}_+, \quad \text{hence} \quad [2\rho, e^{-\Delta_k/4}] = \Delta_k e^{-\Delta_k/4}.$$

For arbitrary $q \in \mathcal{P}$ and $p := e^{\Delta_k/4}q$ it now follows that

$$2\rho(q) \,=\, (2\rho e^{-\Delta_k/4})(p) \,=\, 2e^{-\Delta_k/4}\rho(p) + \Delta_k e^{-\Delta_k/4}p \,=\, 2e^{-\Delta_k/4}\rho(p) + \Delta_k q.$$

Hence for $a \in \mathbb{C}$, there are equivalent:

$$(\Delta_k - 2\rho)(q) = -2aq \iff \rho(p) = ap \iff a = n \in \mathbb{Z}_+ \text{ and } p \in \mathcal{P}_n.$$

This yields the assertion.

(2) We first verify by induction that

$$\big[\Delta_k, E^n\big] = 2nE^{n-1}H + 2n(n-1)E^{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

In case $n = 1$ this is clear from (3.4); if $n \geq 1$, then

$$\big[\Delta_k, E^{n+1}\big] = \big[\Delta_k, E^n\big]\, E + E^n \big[\Delta_k, E\big] = 2nE^{n-1}HE + 2n(n-1)E^n + 2E^nH,$$

by our induction hypothesis. Using the identity $HE = EH + 2E$, we readily see that (3.5) holds for $n+1$ as well.

From (3.5) it is now easily deduced that $\big[\Delta_k, e^{-E}\big] = -2e^{-E}H + 2Ee^{-E}$. We thus obtain

$$(\Delta_k - |x|^2)f \,=\, \Delta_k\big(e^{-E}q\big) - 2Ee^{-E}q \,=\, e^{-E}\Delta_k q - 2e^{-E}\,(\rho + \gamma + N/2)q\,. \tag{3.6}$$

The stated relation is now a consequence of (1).

□

3.5. Corollary. (i) *The generalized Hermite polynomials satisfy the following differential- difference equation:*

$$\Big(\Delta_k - 2\sum_{i=1}^N x_i\partial_i\Big)H_\nu \,=\, -2|\nu|H_\nu\,, \qquad \nu \in \mathbb{Z}_+^N.$$

(ii) *The generalized Hermite functions $\{h_\nu\,, \nu \in \mathbb{Z}_+^N\}$ form a complete set of eigenfunctions for the operator $\Delta_k - |x|^2$ on $L^2(\mathbb{R}^N, w_k(x)dx)$ with*

$$\big(\Delta_k - |x|^2\big)\, h_\nu \,=\, -(2|\nu| + 2\gamma + N)\, h_\nu\,.$$

Note also that as a consequence of the above theorem, the operator $\Delta_k - 2\rho$ has for each $p \in \mathcal{P}_n$ a unique polynomial eigenfunction q of the form $q = p + r$, where the degree of r is less than n; it is given by $q = e^{-\Delta_k/4}p$.

3.6. Examples. (3) **The S_N-case.** For the symmetric group $G = S_N$ (acting on $\mathbb{R}^N$ by permuting the coordinates), the mulitplicity function is characterized by a single parameter which is often denoted by $1/\alpha > 0$, and the corresponding weight function is given by $w_S(x) = \prod_{i<j}|x_i - x_j|^{2/\alpha}$. The associated Dunkl operators are

$$T_i^S \,=\, \partial_i + \frac{1}{\alpha}\sum_{j\neq i}\frac{1 - s_{ij}}{x_i - x_j} \quad (i = 1, \dots, N),$$

where s_{ij} denotes the operator transposing x_i and x_j. The operator $\Delta_S - 2\rho$ is a Schrödinger operator of Calogero-Sutherland type, involving an external harmonic potential with exchange terms, see [B-F2] and [B-F3]. It is given explicitely by

$$\Delta_S - 2\rho \;=\; \Delta - 2\sum_{i=1}^{N} x_i\partial_i + \frac{2}{\alpha}\sum_{i<j}\frac{1}{x_i - x_j}\Big[(\partial_i - \partial_j) - \frac{1 - s_{ij}}{x_i - x_j}\Big]. \tag{3.7}$$

In [B-F2], Baker and Forrester study "non-symmetric generalized Hermite polynomials" $E_\nu^{(H)}$, which they define as the unique eigenfunctions of (3.7) of the form

$$E_\nu^{(H)} \;=\; E_\nu + \sum_{|\mu|<|\nu|} c_{\mu,\nu} E_\mu\,,$$

where the E_ν, $\nu \in \mathbb{Z}_+^N$ are the non-symmetric Jack polynomials (associated with S_N and α) as defined e.g. in [Op2] (see also [K-S]). Thus $E_\nu^{(H)} = e^{-\Delta_S/4}E_\nu$ (just by Lemma 3.4), and indeed, up to some normalization factors, the $E_\nu^{(H)}$ make up a system of generalized Hermite polynomials for S_N in our sense. This follows from the fact that the non-symmetric Jack polynomials E_ν, being homogeneous of degree $|\nu|$ and forming a vector space basis of $\mathcal{P}$, are also orthogonal with respect to the inner product $[.\,,.]_S$. This was proved in [B-F3] via orthogonality of the $E_\nu^{(H)}$. A short direct proof can be given as follows: According to [Op2], Prop. 2.10, the E_ν are simultaneous eigenfunctions of the Cherednik operators ξ_i for S_N, which were introduced in [Che] and can be written as

$$\xi_i = \alpha x_i T_i^S + 1 - N + \sum_{j>i} s_{ij} \qquad (i = 1,\dots,N). \tag{3.8}$$

In fact, the E_ν satisfy $\xi_i E_\nu = \overline{\nu}_i E_\nu$, where the eigenvalues $\overline{\nu} = (\overline{\nu}_1,\dots,\overline{\nu}_N)$ are given explicitely in [Op2]. They are distinct, i.e. if $\nu \neq \mu$, then $\overline{\nu} \neq \overline{\mu}$. On the other hand, (3.8) shows immediately that the operators ξ_i are symmetric with respect to $[.\,,.]_S$. (For all $p,q \in \mathcal{P}$ and $g \in S_N$, we have $[g(p), g(q)]_S = [p,q]_S$ for $g(p)(x) = p(g^{-1}(x))$, as noted in [D-dJ-O].) Together, this proves that the E_ν are orthogonal with respect to $[.\,,.]_S$. Hence a possible choice for the basis $\{\varphi_\nu\}$ is to set $\varphi_\nu = d_\nu E_\nu$, with some normalization constants $d_\nu > 0$.

We finally remark that in this case the locally uniform convergence of the series in Lemma 3.1(ii) extends to $\mathbb{C}^N \times \mathbb{C}^N$, see also [B-F3], Prop. 3.10. This is because the coefficients of the non-symmetric Jack-polynomials E_ν in their monomial expansions are known to be nonnegative ([K-S], Theorem 4.11), hence $|E_\nu(z)| \leq E_\nu(|z|)$ for all $z \in \mathbb{C}^N$.

(4) **A remark on the B_N-case.** Suppose that G is the Weyl group of type B_N, generated by sign-changes and permutations. Here the multiplicity function is characterized by two parameters $k_0,\, k_1 \geq 0$. The weight function is

$$w_B(x) = \prod_{i=1}^{N} |x_i|^{2k_1} \prod_{i<j} |x_i^2 - x_j^2|^{2k_0}.$$

Let T_i^B and Δ_B denote the associated Dunkl operators and Laplacian. We consider the space

$$W := \{f \in C^1(\mathbb{R}^N) : \, f(x) = F(x^2) \quad \text{for some } F \in C^1(\mathbb{R}^N)\}$$

of "completely even" functions; here $x^2 = (x_1^2, \ldots, x_N^2)$. It is easily checked that for completely even f, $\Delta_B f$ is also completely even. The restriction of Δ_B to W is given by

$$\begin{aligned}\Delta_B|_W \; &= \Delta + 2k_1 \sum_{i=1}^N \frac{1}{x_i}\partial_i \, + \, 2k_0 \sum_{i<j}\Big(\frac{1}{x_i - x_j}(\partial_i - \partial_j) + \frac{1}{x_i + x_j}(\partial_i + \partial_j)\Big) \\ &\quad - 2k_0 \sum_{i<j}\Big(\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2}\Big)(1 - s_{ij}).\end{aligned}$$

Again, the operator $(\Delta_B - 2\rho)|_W$ is of Calogero-Sutherland type. Its completely even polynomial eigenfunctions are discussed in [B-F2] and [B-F3] separately from the Hermite-case; they are called "non-symmetric Laguerre polynomials" and denoted by $E_\nu^{(L)}(x^2)$. It is easy to see that they make up the completely even subsystem of a suitably choosen generalized Hermite-system $\{H_\nu\}$ for B_N (and parameters k_0, k_1, where we assume $k_0 > 0$):

To this end, let again E_ν denote the S_N-type non-symmetric Jack polynomials, corresponding to $\alpha = 1/k_0$. For $\nu \in \mathbb{Z}_+^N$ set $\widehat{E}_\nu(x) := E_\nu(x^2)$. These modified Jack polynomials form a basis of $\mathcal{P} \cap W$. The non-symmetric Laguerre polynomials of Baker and Forrester can be written as

$$E_\nu^{(L)}(x^2) \, = \, e^{-\Delta_B/4}\widehat{E}_\nu(x)\,.$$

(Note that the polynomials on the right side are in fact completely even and eigenfunctions of $\Delta_B - 2\rho$.) Involving again the S_N-type Cherednik operators from (3), it is easily checked that the $\widehat{E}_\nu$ are orthogonal with respect to the scalar product $[.\,,.]_B$: The ξ_i induce operators $\widehat{\xi}_i \; (i = 1, \ldots, N)$ on W by

$$\widehat{\xi}_i f(x) := (\xi_i F)(x^2) \quad \text{if } \; f(x) = F(x^2),$$

c.f. [B-F3]. Thus $\widehat{\xi}_i \widehat{E}_\nu = \overline{\nu}_i \widehat{E}_\nu$, and a short calculation gives

$$\widehat{\xi}_i f(x) \, = \, \alpha x_i^2 (T_i^S F)(x^2) + 1 - N + \sum_{j>i} s_{ij} \, = \, \frac{\alpha}{2} x_i T_i^B f(x) + 1 - N + \sum_{j>i} s_{ij}.$$

This proves that the $\widehat{\xi}_i$ are symmetric with respect to $[.\,,.]_B$ and yields our assertion by the same argument as in the previous example. We therefore obtain an orthonormal basis $\{\varphi_\nu\}$ of $\mathcal{P}$ with respect to $[.\,,.]_B$ by setting $\varphi_\nu := d_\nu \widehat{E}_\eta$ for $\nu = (2\eta_1, \ldots, 2\eta_N)$ and completing the set $\{\varphi_\nu \,, \; \nu \in (2\mathbb{Z}_+)^N\}$ by a Gram-Schmidt procedure.

Many properties of the classical Hermite polynomials and Hermite functions on $\mathbb{R}^N$ have natural extensions to our general setting. We start with a Rodrigues-formula:

3.7. Theorem. *For all* $\nu \in \mathbb{Z}_+^N$ *and* $x \in \mathbb{R}^N$ *we have*

$$H_\nu(x) = (-1)^{|\nu|} e^{|x|^2} \varphi_\nu(T)\, e^{-|x|^2}. \tag{3.9}$$

Proof. First note that if p is a polynomial of degree $n \geq 0$, then

$$p(T)\, e^{-|x|^2} = q(x)\, e^{-|x|^2}$$

with a polynomial q of the same degree. This follows easily from induction by the degree of p, together with the product rule (2.1). In particular, the function

$$Q_\nu(x) := (-1)^{|\nu|} e^{|x|^2} \varphi_\nu(T)\, e^{-|x|^2} = e^{|x|^2} \varphi_\nu(-T)\, e^{-|x|^2}$$

is a polynomial of degree $|\nu|$. In order to prove that $Q_\nu = H_\nu$, it therefore suffices to show that for each $\eta \in \mathbb{Z}_+^N$ with $|\eta| \leq |\nu|$,

$$2^{-|\eta|} \int_{\mathbb{R}^N} Q_\nu(x) H_\eta(x)\, d\mu_k(x) \;=\; \delta_{\nu,\eta}\,, \tag{3.10}$$

where $\delta_{\nu,\eta}$ denotes the Kronecker delta. Using the antisymmetry of the T_i with respect to $L^2(\mathbb{R}^N, w_k(x)dx)$ (Lemma 2.9 of [D4]) as well as the commutativity of $\{T_i\}$, we can write

$$\begin{aligned}
2^{-|\eta|} \int_{\mathbb{R}^N} Q_\nu(x) H_\eta(x)\, d\mu_k(x) &= c_k \int_{\mathbb{R}^N} \varphi_\nu(-T)\big(e^{-|x|^2}\big) e^{-\Delta_k/4} \varphi_\eta(x)\, w_k(x) dx \\
&= c_k \int_{\mathbb{R}^N} e^{-|x|^2} \big(\varphi_\nu(T) e^{-\Delta_k/4} \varphi_\eta\big)(x)\, w_k(x) dx = \int_{\mathbb{R}^N} \big(e^{-\Delta_k/4} \varphi_\nu(T)\, \varphi_\eta\big)(x)\, d\mu_k(x).
\end{aligned}$$

But as $|\eta| \leq |\nu|$, we have $\varphi_\nu(T)\, \varphi_\eta = [\varphi_\nu, \varphi_\eta]_k = \delta_{\nu,\eta}$ from which (3.10) follows. □

There is also a generating function for the generalized Hermite polynomials:

3.8. Proposition. *For* $n \in \mathbb{Z}_+$ *and* $z, w \in \mathbb{C}^N$ *put* $L_n(z,w) := \sum_{|\nu|=n} H_\nu(z)\, \varphi_\nu(w)$. *Then*

$$\sum_{n=0}^{\infty} L_n(z,w) \;=\; e^{-l(w)} K(2z, w),$$

the convergence of the series being normal on $\mathbb{C}^N \times \mathbb{C}^N$.

Proof. Suppose first that $z, w \in \mathbb{R}^N$. By definition of the H_ν and in view of formula (2.6) for K_n we may write

$$\begin{aligned}
L_n(z,w) = 2^n e^{-\Delta_k^z/4} K_n(z,w) &= 2^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{4^m m!} l(w)^m K_{n-2m}(z,w) \\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!} l(w)^m K_{n-2m}(2z,w)
\end{aligned}$$

for all $n \in \mathbb{Z}_+$. By analytic continuation, this holds for all $z, w \in \mathbb{C}^N$ as well. Using estimation (2.4), one obtains

$$S_n(z,w) := \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{m!}\, |l(w)|^m |K_{n-2m}(2z,w)| \;\leq\; \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{m!}\, |w|^{2m} \cdot \frac{|2z|^{n-2m} |w|^{n-2m}}{(n-2m)!}.$$

If n is even, set $k := n/2$ and estimate further as follows:

$$S_n(z,w) \leq \frac{|w|^{2k}}{k!} \sum_{m=0}^{k} \binom{k}{m} (2|z|^2)^{k-m} \;=\; \frac{1}{k!}\big(|w|^2(1+2|z|^2)\big)^k.$$

A similar estimation holds if n is odd. This entails the normal convergence of the series $\sum_{n=0}^{\infty} L_n(z,w)$ on $\mathbb{C}^N \times \mathbb{C}^N$, and also that

$$\begin{aligned}\sum_{n=0}^{\infty} L_n(z,w) \;&=\; \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, l(w)^m K_{n-2m}(2z,w) \quad (\text{with} \;\; K_j := 0 \;\text{for}\;\; j<0)\\ &=\; \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\, l(w)^m \sum_{n=0}^{\infty} K_{n-2m}(2z,w) \;=\; e^{-l(w)} K(2z,w)\end{aligned}$$

for all $z, w \in \mathbb{C}^N$. $\square$

Applying Lemma 2.1 to $p = \varphi_\nu$ with $c = -1/4$ and $a = 1/\lambda$, we obtain the following formula for the generalized Hermite polynomials:

3.9. Lemma. *For $\lambda \in \mathbb{C} \setminus \{0\}$, $\nu \in \mathbb{Z}_+^N$ and $x \in \mathbb{R}^N$,*

$$\Big(\frac{\lambda}{2}\Big)^{|\nu|} H_\nu\Big(\frac{x}{\lambda}\Big) \;=\; \big(e^{-\lambda^2 \Delta_k/4}\varphi_\nu\big)(x).$$

3.10. Proposition. *The generalized Hermite functions $\{h_\nu,\, \nu \in \mathbb{Z}_+^N\}$ are a basis of eigenfunctions of the Dunkl transform $\mathcal{D}_k$ on $L^2(\mathbb{R}^N, w_k(x)dx)$, satisfying*

$$\mathcal{D}_k(h_\nu) \;=\; 2^{\gamma+N/2} c_k^{-1}\, (-i)^{|\nu|}\, h_\nu\,.$$

Proof. We use Prop. 2.1 form [D4], which says that for all $p \in \mathcal{P}$ and $z \in \mathbb{C}^N$,

$$\frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x)\, K(x,z)\, w_k(x) e^{-|x|^2/2} dx \;=\; e^{l(z)/2} p(z). \tag{3.11}$$

Here again, $l(z) = \sum_{i=1}^N z_i^2$. Let $p_\nu(x) := e^{\Delta_k/2} H_\nu(x)$. In view of (3.11) we can write

$$\mathcal{D}_k(h_\nu)(\xi) = \int_{\mathbb{R}^N} H_\nu(x) K(-i\xi, x)\, w_k(x) e^{-|x|^2/2} dx \;=\; 2^{\gamma+N/2} c_k^{-1}\, e^{-|\xi|^2/2} p_\nu(-i\xi)$$

for all $\xi \in \mathbb{R}^N$. By definition of H_ν we have $p_\nu(x) = 2^{|\nu|} e^{\Delta_k/4}\varphi_\nu(x)$. So we arrive at

$$\mathcal{D}_k(h_\nu)(\xi) = 2^{\gamma+N/2} c_k^{-1}\, e^{-|\xi|^2/2}\, 2^{|\nu|} \big(e^{\Delta_k/4}\varphi_\nu\big)(-i\xi).$$

Application of Lemma 3.9 with $\lambda = -i$ now yields that $\big(e^{\Delta_k/4}\varphi_\nu\big)(-i\xi) = (-i/2)^{|\nu|} H_\nu(\xi)$, hence

$$\mathcal{D}_k(h_\nu)(\xi) \;=\; 2^{\gamma+N/2} c_k^{-1} (-i)^{|\nu|} h_\nu(\xi).$$

$\square$

We finish this section with a Mehler-type formula for the generalized Hermite polynomials. For this, we need the following integral representation:

3.11. Lemma. *For all $x, y \in \mathbb{R}^N$ and $\nu \in \mathbb{Z}_+^N$ we have*

$$e^{-|x|^2} H_\nu(x) \;=\; 2^{|\nu|} \int_{\mathbb{R}^N} K(x, -2iy)\, \varphi_\nu(iy)\, d\mu_k(y).$$

Proof. A short calculation, using again relation (2.2), shows that for homogeneous polynomials p, formula (3.11) may be rewritten as

$$\int_{\mathbb{R}^N} e^{-\Delta_k/4} p(x)\, K(x, 2z)\, d\mu_k(x) \;=\; e^{l(z)} p(z) \quad (z \in \mathbb{C}^N). \tag{3.12}$$

By linearity, this holds for all $p \in \mathcal{P}$. Lemma 3.9 with $\lambda = i$ further shows that

$$e^{\Delta_k/4} \varphi_\nu(x) = \big(\frac{i}{2}\big)^{|\nu|} H_\nu(-ix).$$

As φ_ν is homogeneous of degree $|\nu|$, we thus can write $\varphi_\nu(2iy) \;=\; (-i)^{|\nu|}\, e^{-\Delta_k/4} H_\nu^*(y)$ with $H_\nu^*(y) = H_\nu(iy)$. From (3.12) it now follows that

$$\int_{\mathbb{R}^N} K(x, -2iy)\, \varphi_\nu(2iy)\, d\mu_k(y) \;=\; e^{-|x|^2} H_\nu^*(-ix),$$

which yields the assertion. □

3.12. Theorem. (Mehler-formula for the H_ν) *For $r \in \mathbb{C}$ with $|r| < 1$ and all $x, y \in \mathbb{R}^N$,*

$$\sum_{\nu \in \mathbb{Z}_+^N} \frac{H_\nu(x) H_\nu(y)}{2^{|\nu|}}\, r^{|\nu|} \;=\; \frac{1}{(1-r^2)^{\gamma + N/2}} \exp\Big\{ -\frac{r^2(|x|^2 + |y|^2)}{1 - r^2} \Big\}\, K\Big(\frac{2rx}{1-r^2}, \, y\Big).$$

Proof. Consider the integral

$$M(x, y, r) := c_k^2 \int_{\mathbb{R}^N \times \mathbb{R}^N} K(-2rz, v) K(-2iz, x) K(-2iv, y)\, w_k(z) w_k(v)\, e^{-(|z|^2 + |v|^2)} d\,(z, v).$$

The bounds (2.4) and (2.5) on K assure that it converges for all $r \in \mathbb{C}$ with $|r| < 1$ and all $x, y \in \mathbb{R}^N$. Now write $K(-2rz, v) = \sum_{n=0}^\infty (2r)^n K_n(iz, iv)$ in the integral above. As

$$\sum_{n=0}^\infty |2r|^n |K_n(iz, iv)| \leq e^{2|r||z||v|},$$

the dominated convergence theorem yields that

$$\begin{aligned} M(x, y, r) &= \sum_{n=0}^\infty (2r)^n \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_n(iz, iv)\, K(-2iz, x)\, K(-2iv, y)\, d\mu_k(z)\, d\mu_k(v) \\ &= \sum_{n=0}^\infty (2r)^n \sum_{|\nu| = n} \Big(\int_{\mathbb{R}^N} K(-2iz, x)\, \varphi_\nu(iz)\, d\mu_k(z) \Big) \Big(\int_{\mathbb{R}^N} K(-2iv, y)\, \varphi_\nu(iv)\, d\mu_k(v) \Big). \end{aligned}$$

From the above lemma we thus obtain

$$M(x, y, r) = \; e^{-(|x|^2 + |y|^2)} \sum_{\nu \in \mathbb{Z}_+^N} r^{|\nu|} \frac{H_\nu(x) H_\nu(y)}{2^{|\nu|}}, \tag{3.13}$$

and this series, as a power series in r, converges absolutely for all $x, y \in \mathbb{R}^N$. On the other hand, iterated integration and repeated application of formula (2.3) and the reproducing formula (2.8) show that for real r with $|r| < 1$ we have

$$\begin{aligned}
M(x,y,r) &= c_k \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} K(-2rz, v)\, K(-2iy, v)\, d\mu_k(v) \Big) K(-2iz, x) e^{-|z|^2} w_k(z) dz \\
&= c_k\, e^{-|y|^2} \int_{\mathbb{R}^N} e^{(r^2-1)|z|^2} K(2iry, z)\, K(-2ix, z)\, w_k(z)\, dz \\
&= c_k (1-r^2)^{-(\gamma+N/2)} e^{-|y|^2} \int_{\mathbb{R}^N} e^{-|u|^2} K\Big(u\,, \frac{2iry}{\sqrt{1-r^2}}\Big) K\Big(u\,, \frac{-2ix}{\sqrt{1-r^2}}\Big) w_k(u)\, du \\
&= (1-r^2)^{-(\gamma+N/2)} \exp\Big\{ -\frac{|x|^2+|y|^2}{1-r^2} \Big\}\, K\Big(\frac{2rx}{1-r^2}, y\Big).
\end{aligned}$$

By analytic continuation, this holds for $\{r \in \mathbb{C} : |r| < 1\}$ as well. Together with (3.13), this finishes the proof. $\square$

4 The heat equation for Dunkl operators

As before, let Δ_k denote the generalized Laplacian associated with some finite reflection group G on $\mathbb{R}^N$ and a multiplicity function $k \geq 0$ on its root system R. Recall that its action on $C^2(\mathbb{R}^N)$ is given by

$$\Delta_k f \;=\; \Delta f + 2 \sum_{\alpha \in R_+} k(\alpha)\, \delta_\alpha f\,,$$

where

$$\delta_\alpha f(x) \;=\; \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}\,.$$

Its action may as well be restricted to $C^2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is open and invariant under the group operation of G. We call a function $f \in C^2(\Omega)$ k-subharmonic on Ω, if $\Delta_k f \geq 0$ on Ω.

The generalized Laplacian satisfies the following maximum principle, which will be important later on:

4.1. Lemma. *Let $\Omega \subseteq \mathbb{R}^N$ be open and G-invariant. If a real-valued function $f \in C^2(\Omega)$ attains an absolute maximum at $x_0 \in \Omega$, i.e. $f(x_0) = \sup_{x\in\Omega} f(x)$, then*

$$\Delta_k f(x_0) \;\leq\; 0\,.$$

Proof. Let $D^2 f(x)$ denote the Hessian of u in $x \in \Omega$. The given situation enforces that $\nabla f(x_0) = 0$ and $D^2 f(x_0)$ is negative semi-definite; in particular, $\Delta f(x_0) \leq 0$. Moreover, $f(x_0) \geq f(\sigma_\alpha x_0)$ for all $\alpha \in R$, so the statement is obvious in the case that $\langle \alpha, x_0 \rangle \neq 0$ for all $\alpha \in R$. If $\langle \alpha, x_0 \rangle = 0$ for some $\alpha \in R$, we have to argue more carefully: Choose an open ball $B \subseteq \Omega$ with center x_0. Then $\sigma_\alpha x \in B$ for $x \in B$, and $\sigma_\alpha x - x = -\langle \alpha, x \rangle \alpha$. Now Taylor's formula yields

$$f(\sigma_\alpha x) - f(x) = -\langle \alpha, x \rangle\, \langle \nabla f(x), \alpha \rangle + \frac{1}{2} \langle \alpha, x \rangle^2\, \alpha^t D^2 f(\xi) \alpha\,,$$

with some ξ on the line segment between x and $\sigma_\alpha x$. It follows that for $x \in B$ with $\langle \alpha, x\rangle \neq 0$ we have $\delta_\alpha f(x) = \frac{1}{2}\alpha^t D^2 u(\xi)\alpha$. Passing to the limit $x \to x_0$ now leads to $\delta_\alpha f(x_0) = \frac{1}{2}\alpha^t D^2 f(x_0)\alpha \leq 0$, which finishes the proof. $\square$

At this stage it is not much effort to gain a weak maximum principle for k-subharmonic functions on bounded, G-invariant subsets of $\mathbb{R}^N$, which we want to include here before passing over to the heat equation. Its range of validity is quite general, in contrast to the strong maximum principle in [D1], which is restricted to k-harmonic polynomials on the unit ball. Our proof follows the classical one for the usual Laplacian, as it can be found e.g. in [J].

4.2. Theorem. *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and G-invariant, and let $f \in C^2(\Omega) \cap C(\overline{\Omega})$ be real-valued and k-subharmonic on Ω. Then*

$$\max_{\overline{\Omega}}(f) \,=\, \max_{\partial\Omega}(f)\,.$$

Proof. Fix $\epsilon > 0$ and put $g := f + \epsilon|x|^2$. A short calculation gives $\Delta_k(|x|^2) = 2N + 4\gamma > 0$. Hence $\Delta_k g > 0$ on Ω, and Lemma 4.1 shows that g cannot achieve its maximum on $\overline{\Omega}$ at any $x_0 \in \Omega$. It follows that

$$\max_{\overline{\Omega}}\,(f + \epsilon|x|^2) \,=\, \max_{\partial\Omega}\,(f + \epsilon|x|^2)$$

for each $\epsilon > 0$. Consequently,

$$\max_{\overline{\Omega}}(f) \,+\, \epsilon \min_{\overline{\Omega}}|x|^2 \,\leq\, \max_{\partial\Omega}(f) \,+\, \epsilon \max_{\partial\Omega}|x|^2.$$

The assertion now follows with $\epsilon \to 0$. $\square$

In this section we consider the generalized heat operator

$$H_k \,:=\, \Delta_k - \partial_t$$

on function spaces $C^2(\Omega \times (0,T))$, where $T > 0$ and $\Omega \subseteq \mathbb{R}^N$ is open and G-invariant. Among the variety of initial- and boundary value problems which may be posed for H_k in analogy to the corresponding classical problems, we here focus on the homogeneous Cauchy problem:

Find $u \in C^2(\mathbb{R}^N \times (0,T))$ which is continuous on $\mathbb{R}^N \times [0,T]$ and satisfies

$$\begin{cases} H_k u = 0 & \text{on } \mathbb{R}^N \times (0,T), \\ u(.\,,0) = f & \in C_b(\mathbb{R}^N). \end{cases} \tag{4.1}$$

First of all, let us note some basic solutions of the generalized heat equation $H_k u = 0$. Again we set $\gamma := \sum_{\alpha \in R_+} k(\alpha) \,\geq 0$.

4.3. Lemma. *For parameters $a \geq 0$ and $b \in \mathbb{R} \setminus \{0\}$, the function*

$$u(x,t) = \frac{1}{(a - bt)^{\gamma + N/2}}\,\exp\Big\{\frac{b|x|^2}{4(a - bt)}\Big\}$$

solves $H_k u = 0$ on $\mathbb{R}^N \times (-\infty, a/b)$ in case $b > 0$, and on $\mathbb{R}^N \times (a/b, \infty)$ in case $b < 0$.

Proof. The product rule (2.1) together with $\sum_{i=1}^N T_i x_i = N + 2\gamma$ shows that for each $\lambda > 0$,

$$\Delta_k\big(e^{\lambda|x|^2}\big) = \sum_{i=1}^N T_i\big(2\lambda x_i\, e^{\lambda|x|^2}\big) = 2\lambda\,(N + 2\gamma + 2\lambda|x|^2)\, e^{\lambda|x|^2}.$$

From this the statement is obtained readily by a short calculation. $\square$

In particular, the function

$$F_k(x,t) = \frac{M_k}{t^{\gamma+N/2}} e^{-|x|^2/4t}, \quad \text{with} \quad M_k = 4^{-\gamma-N/2} c_k\,,$$

is a solution of the heat equation $H_k u = 0$ on $\mathbb{R}^N \times (0,\infty)$. It generalizes the fundamental solution for the classical heat equation $\Delta u - \partial_t u = 0$, which is given by $F_0(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$. The normalization constant M_k is choosen such that

$$\int_{\mathbb{R}^N} F_k(x,t)\, w_k(x) dx \;=\; 1 \quad \text{for all } t > 0.$$

In order to solve the Cauchy problem (4.1), it suggests itself to apply Fourier transform methods – in our case, the Dunkl transform – under suitable decay assumptions on the initial data f. In fact, in the classical case $k = 0$ a bounded solution of (4.1) is obtained by convolving f with the fundamental solution F_0, and its uniqueness is a consequence of a well-known maximum principle for the heat operator. It is not much effort to extend this maximum principle to the generalized heat operator H_k in order to obtain uniqeness results; we shall do this in Prop. 4.12 and Theorem 4.13 at the end of this section. However, in our general situation it is not known whether there exists a reasonable convolution structure on $\mathbb{R}^N$ matching the action of the Dunkl transform $\mathcal{D}_k$, i.e. making it a homomorphism on suitable function spaces. In the one-dimensional case this is true: there is a L^1-convolution algebra associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ and the multiplicity parameter $k = \mu \geq 0$; this convolution enjoys many properties of a group convolution. It is studied in [Rö] (see also [Rö-V] and [Ros]).

In the N-dimensional case, we may introduce the notion of a generalized translation at least on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ (and similar on $L^2(\mathbb{R}^N, w_k(x)dx)$, as follows:

$$L_k^y f(x) := \frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N} \mathcal{D}_k f(\xi)\, K(ix,\xi) K(iy,\xi)\, w_k(\xi) d\xi; \quad y \in \mathbb{R}^N,\; f \in \mathcal{S}(\mathbb{R}^N). \tag{4.2}$$

Note that in case $k = 0$, we simply have $L_0^y f(x) = f(x+y)$, while in the one-dimensional case, (4.2) matches the above-mentioned convolution structure on $\mathbb{R}$. Clearly, $L_k^y f(x) = L_k^x f(y)$; moreover, the inversion theorem for the Dunkl transform assures that $L_k^y f = f$ for $y = 0$ and $\mathcal{D}_k(L_k^y f)(\xi) = K(iy,\xi)\mathcal{D}_k f(\xi)$. From this it is not hard to see (by use of the bounds (2.7)) that $L_k^y f$ belongs to $\mathcal{S}(\mathbb{R}^N)$ again.

Let us now consider the "fundamental solution" $F_k(.\,,t)$ for $t > 0$. A short calculation, using Prop. 3.10 or Lemma 4.11 of [dJ], shows that

$$(\mathcal{D}_k F_k)(\xi,t) \;=\; e^{-t|\xi|^2}. \tag{4.3}$$

By use of the reproducing formula (2.8) one therefore obtains from (4.2) the representation

$$L_k^{-y}F_k(x,t) \;=\; \frac{M_k}{t^{\gamma+N/2}}e^{-(|x|^2+|y|^2)/4t}\,K\Big(\frac{x}{\sqrt{2t}},\frac{y}{\sqrt{2t}}\Big). \tag{4.4}$$

4.4. Definition. The generalized heat kernel Γ_k is given by

$$\Gamma_k(x,y,t) := \frac{M_k}{t^{\gamma+N/2}}\,e^{-(|x|^2+|y|^2)/4t}\,K\Big(\frac{x}{\sqrt{2t}},\frac{y}{\sqrt{2t}}\Big), \quad x,y\in\mathbb{R}^N,\; t>0.$$

4.5. Lemma. *The heat kernel Γ_k has the following properties on $\mathbb{R}^N\times\mathbb{R}^N\times(0,\infty)$:*

(1) $\displaystyle \Gamma_k(x,y,t) \;=\; \frac{c_k^2}{4^{\gamma+N/2}}\int_{\mathbb{R}^N} e^{-t|\xi|^2}\,K(ix,\xi)\,K(-iy,\xi)\,w_k(\xi)d\xi\,.$

(2) *For fixed $y\in\mathbb{R}^N$, the function $u(x,t) := \Gamma_k(x,y,t)$ solves the generalized heat equation $H_ku=0$ on $\mathbb{R}^N\times(0,\infty)$.*

(3) $\displaystyle \int_{\mathbb{R}^N}\Gamma_k(x,y,t)\,w_k(x)dx \;=\; 1.$

(4) $\displaystyle |\Gamma_k(x,y,t)| \;\le\; \frac{M_k}{t^{\gamma+N/2}}\,e^{-(|x|-|y|)^2/4t}.$

Proof. (1) is clear from the above derivation. For (2), remember that $\Delta_k^x\,K(ix,\xi) = -|\xi|^2K(ix,\xi)$. Hence the assertion follows at once from representation (1) by taking the differentiations under the integral. This is justified by the decay properties of the integrand and its derivatives in question (use estimation (2.7) for the partial derivatives of $K(ix,\xi)$ with respect to x.) To obtain (3), we employ formula (4.4) as well as (4.3) and write

$$\int_{\mathbb{R}^N}\Gamma_k(x,y,t)\,w_k(x)dx \;=\; \mathcal{D}_k\big(L_k^{-y}F_k\big)(0,t) \;=\; K(-iy,0)(\mathcal{D}_kF_k)(0,t) \;=\; 1.$$

Finally, (4) is a a consequence of the estimate (2.4) for K. □

Remark. In contrast to the classical case, it is not yet clear at this stage that the kernel Γ_k is generally nonnegative. In fact, it is still an open conjecture that the function $K(iy,.)$ is positive-definite on $\mathbb{R}^N$ for each $y\in\mathbb{R}^N$ (c.f. the remarks in [dJ] and [D3]). This would imply a Bochner-type integral representation of $K(iy,.)$ and positivity of K on $\mathbb{R}^N\times\mathbb{R}^N$ as an immediate consequence. In the one-dimensional case this conjecture is true, and the Bochner-type integral representation is explicitely known (see [Ros] or [Rö]). By one-parameter semigroup techniques, it will however soon turn out that K is at least positive on $\mathbb{R}^N\times\mathbb{R}^N$.

4.6. Definition. For $f\in C_b(\mathbb{R}^N)$ and $t\ge 0$ set

$$H(t)f(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^N}\Gamma_k(x,y,t)f(y)\,w_k(y)dy & \text{if } t>0,\\ f(x) & \text{if } t=0\end{cases} \tag{4.5}$$

Part (4) of Lemma 4.5 assures that for each $t \geq 0$, $H(t)f$ is well-defined and continuous on $\mathbb{R}^N$. It provides a natural candidate for the solution to our Cauchy problem. Indeed, when restricting to initial data from the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, we easily obtain the following:

4.7. Theorem. *Suppose that* $f \in \mathcal{S}(\mathbb{R}^N)$*. Then* $u(x,t) := H(t)f(x),\; (x,t) \in \mathbb{R}^N \times [0,\infty)$*, solves the Cauchy-problem* (4.1) *for each* $T > 0$*. Morover, it has the following properties:*

(i) $H(t)f \in \mathcal{S}(\mathbb{R}^N)$ *for each* $t > 0$.

(ii) $H(t+s)f \,=\, H(t)H(s)f$ *for all* $s,t \geq 0$.

(iii) $\|H(t)f - f\|_{\infty,\mathbb{R}^N} \to 0$ *with* $t \to 0$.

Proof. Using formula (1) of Lemma 4.5 and Fubini's theorem, we can write

$$
\begin{aligned}
u(x,t) = H(t)f(x) \,&= \frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(ix,\xi)K(-iy,\xi)\, e^{-t|\xi|^2} f(y)\, w_k(\xi)w_k(y)\, d\xi dy \\
&= \frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{-t|\xi|^2}\mathcal{D}_k f(\xi)K(ix,\xi)\, w_k(\xi)d\xi \qquad (4.6)
\end{aligned}
$$

for all $t > 0$. (Remember that $\mathcal{S}(\mathbb{R}^N)$ is invariant under the Dunkl transform). This makes clear that (i) is satisfied. As before, it is seen that differentiations may be taken under the integral in (4.6), and that $H_k u = 0$ on $\mathbb{R}^N \times (0,\infty)$. Moreover, in view of the inversion theorem for the Dunkl transform, (4.6) holds for $t = 0$ as well. Using (2.5), we thus obtain the estimation

$$
\|H(t)f - f\|_{\infty,\mathbb{R}^N} \,\leq\, \sqrt{|G|}\frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N} |\mathcal{D}_k f(\xi)|\big(1 - e^{-t|\xi|^2}\big) w_k(\xi)d\xi,
$$

and this integral tends to 0 with $t \to 0$. This yields (iii). In particular, it follows that u is continuous on $\mathbb{R}^N \times [0,\infty)$. To prove (ii), note that $\mathcal{D}_k\big(H(t)f\big)(\xi) \,=\, e^{-t|\xi|^2}\mathcal{D}_k f(\xi)$. Therefore

$$
\mathcal{D}_k\big(H(t+s)f\big)(\xi) \,=\, e^{-t|\xi|^2}\mathcal{D}_k\big(H(s)f\big)f(\xi) \,=\, \mathcal{D}_k\big(H(t)H(s)f\big)(\xi).
$$

The statement now follows from the injectivity of the Dunkl transform on $\mathcal{S}(\mathbb{R}^N)$. □

We are now going to show that indeed, the linear operators $H(t)$ on $\mathcal{S}(\mathbb{R}^N)$ extend to a positive contraction semigroup on the Banach space $C_0(\mathbb{R}^N)$, equipped with its uniform norm $\|.\|_\infty$. To this end, we consider the generalized Laplacian Δ_k as a densely defined linear operator on $C_0(\mathbb{R}^N)$ with domain $\mathcal{S}(\mathbb{R}^N)$.

4.8. Theorem. (1) *The operator* Δ_k *on* $C_0(\mathbb{R}^N)$ *is closable, and its closure* $\overline{\Delta}_k$ *generates a positive, strongly continuous contraction semigroup* $\{T(t), t \geq 0\}$ *on* $C_0(\mathbb{R}^N)$.

(2) *The action of* $T(t)$ *on* $\mathcal{S}(\mathbb{R}^N)$ *is given by* $T(t)f = H(t)f$.

Proof. (1) We apply a variant of the Lumer-Phillips theorem, which characterizes generators of positive one-parameter contraction semigroups (see e.g. [A], Cor. 1.3). It requires two properties:

(i) The operator Δ_k satisfies the following "dispersivity condition": Suppose that $f \in \mathcal{S}(\mathbb{R}^N)$ is real-valued with $\max\{f(x) : x \in \mathbb{R}^N\} = f(x_0)$. Then $\Delta_k f(x_0) \leq 0$.

(ii) The range of $\lambda I - \Delta_k$ is dense in $C_0(\mathbb{R}^N)$ for some $\lambda > 0$.

Property (i) is an immediate consequence of Lemma 4.1. Condition (ii) is also satisfied, because $\lambda I - \Delta_k$ maps $\mathcal{S}(\mathbb{R}^N)$ onto itself for each $\lambda > 0$; this follows from the fact that the Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^N)$ and $\mathcal{D}_k\big((\lambda I - \Delta_k)f\big)(\xi) = (\lambda + |\xi|^2)\mathcal{D}_k f(\xi)$. The assertion now follows by the above-mentioned theorem.

(2) It is known from semigroup theory that for every $f \in \mathcal{S}(\mathbb{R}^N)$, the function $t \mapsto T(t)f$ is the unique solution of the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = \overline{\Delta}_k u(t) \quad \text{for } t > 0, \\ u(0) = f \end{cases} \tag{4.7}$$

within the class of all (strongly) continuously differentiable functions u on $[0, \infty)$ with values in the Banach space $(C_0(\mathbb{R}^N), \|.\|_\infty)$. By property (i) of Theorem 4.6 we have $H(t)f \in C_0(\mathbb{R}^N)$ for $f \in \mathcal{S}(\mathbb{R}^N)$. Moreover, from representation (4.6) of $H(t)f$ it is readily seen that $t \mapsto H(t)f$ is continuously differentiable on $[0, \infty)$ and solves (4.7). This finishes the proof. □

4.9. Corollary. *The heat kernel Γ_k is strictly positive on $\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$. In particular, the generalized exponential kernel K satisfies*

$$K(x, y) > 0 \quad \textit{for all } x, y \in \mathbb{R}^N.$$

Proof. For any initial distribution $f \in \mathcal{S}(\mathbb{R}^N)$ with $f \geq 0$ the last theorem implies that

$$\int_{\mathbb{R}^N} \Gamma_k(x, y, t) f(y)\, w_k(y) dy \;=\; T(t)f(x) \;\geq\; 0 \quad \text{for all } (x, t) \in \mathbb{R}^N \times [0, \infty).$$

As $y \mapsto \Gamma_k(x, y, t)$ is continuous on $\mathbb{R}^N$ for each fixed $x \in \mathbb{R}^N$ and $t > 0$, it follows that $\Gamma_k(x, y, t) \geq 0$ for all $x, y \in \mathbb{R}^N$ and $t > 0$. Hence K is nonnegative as well. Now recall again the reproducing identity (2.8), which says that

$$e^{(|x|^2 + |y|^2)} K(2x, y) \;=\; c_k \int_{\mathbb{R}^N} K(x, 2z) K(y, 2z)\, w_k(z)\, e^{-|z|^2} dz$$

for all $x, y \in \mathbb{R}^N$. The integrand on the right side is continuous, non-negative and not identically zero (because $K(x, 0)K(y, 0) = 1$). Therefore the integral itself must be strictly positive. □

4.10. Corollary. *The semigroup $\{T(t)\}$ on $C_0(\mathbb{R}^N)$ is given explicitely by*

$$T(t)f \;=\; H(t)f, \;\; f \in C_0(\mathbb{R}^N).$$

Proof. This is clear from part (2) of Theorem 4.8 and the previous corollary, which implies that the operators $H(t)$ are continuous – even contractive – on $C_0(\mathbb{R}^N)$. □

Remark. The generalized Laplacian also leads to a contraction semigroup on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^N, w_k(x)dx)$; this generalizes the results of [Ros] for the one-dimensional case. In fact, let M denote the multiplication operator on $\mathcal{H}$ defined by $Mf(x) = -|x|^2 f(x)$ and with domain $D(M) = \{f \in \mathcal{H} : |x|^2 f(x) \in \mathcal{H}\}$. M is self-adjoint and generates the strongly continuous contraction semigroup $M(t)f(x) = e^{-t|x|^2} f(x)\,(t \geq 0)$ on $\mathcal{H}$. For $f \in \mathcal{S}(\mathbb{R}^N)$, we have the identity $\mathcal{D}_k(\Delta_k f) = M(\mathcal{D}_k f)$. As $\mathcal{S}(\mathbb{R}^N)$ is dense in $D(M)$, this shows that Δ_k has a self-adjoint extension $\widetilde{\Delta}_k$ on $\mathcal{H}$, namely $\widetilde{\Delta}_k = \mathcal{D}_k^{-1} M \mathcal{D}_k$, where here $\mathcal{D}_k$ denotes the Plancherel-extension of the Dunkl transform to $\mathcal{H}$. The domain of $\widetilde{\Delta}_k$ is the Sobolev-type space $D(\widetilde{\Delta}_k) = \{f \in \mathcal{H} : |\xi|^2\, \mathcal{D}_k f(\xi) \in \mathcal{H}\}$. Being unitarily equivalent with M, the opertor $\widetilde{\Delta}_k$ also generates a strongly continuous contraction semigroup $\{\widetilde{T}(t)\}$ on $\mathcal{H}$ which is unitarily equivalent with $\{M(t)\}$; it is given by

$$\widetilde{T}(t)f(x) \;=\; \int_{\mathbb{R}^N} e^{-t|\xi|^2} \mathcal{D}_k f(\xi)\, K(ix,\xi)\, w_k(\xi) d\xi.$$

The knowledge that Γ_k is nonnegative allows also to solve the Cauchy problem (4.1) in its general setting:

4.11. Theorem. *Let $f \in C_b(\mathbb{R}^N)$. Then $u(x,t) := H(t)f(x)$ is bounded on $\mathbb{R}^N \times [0,\infty)$ and solves the Cauchy problem (4.1) for each $T > 0$.*

Proof. In order to see that u is twice continuously differentiable on $\mathbb{R}^N \times (0,\infty)$ with $H_k u = 0$, we only have to make sure that the necessary differentiations of u may be taken under the integral in (4.5). One has to use again the estimations (2.7) for the partial derivatives of K; these provide sufficient decay properties of the derivatives of Γ_k, allowing the necessary differentiations of u under the integral by use of the dominated convergence theorem. Boundedness of u is clear from the positivity and normalization (Lemma 4.5(3)) of Γ_k; in fact, $|u(x,t)| \leq \|f\|_{\infty,\mathbb{R}^N}$ on $\mathbb{R}^N \times [0,\infty)$.

Finally, we have to show that $H(t)f(x) \to f(\xi)$ with $x \to \xi$ and $t \to 0$. We start by the usual method: For fixed $\epsilon > 0$, choose $\delta > 0$ such that $|f(y) - f(\xi)| < \epsilon$ for $|y - \xi| < 2\delta$ and let $M := \|f\|_{\infty,\mathbb{R}^N}$. Keeping in mind the positivity and normalization of Γ_k, we obtain for $|x - \xi| < \delta$ the estimation

$$\begin{aligned}
|H(t)f(x) - f(\xi)| \;&\leq\; \Big| \int_{\mathbb{R}^N} \Gamma_k(x,y,t)\big(f(y) - f(\xi)\big) w_k(y) dy \Big| \\
&\leq \int_{|y-x|<\delta} \Gamma_k(x,y,t)|f(y) - f(\xi)| w_k(y) dy \;+\; \int_{|y-x|>\delta} \Gamma_k(x,y,t)|f(y) - f(\xi)| w_k(y) dy \\
&< \epsilon + 2M \int_{|y-x|>\delta} \Gamma_k(x,y,t) w_k(y) dy.
\end{aligned}$$

It thus remains to show that for each $\delta > 0$,

$$\lim_{(x,t)\to(\xi,0)} \int_{|y-x|>\delta} \Gamma_k(x,y,t) w_k(y) dy \;=\; 0.$$

For abbreviation put

$$I(x,t) := \int_{|y-x|\leq\delta} \Gamma_k(x,y,t) w_k(y) dy.$$

As $I(x,t) \le 1$, it suffices to prove that $\liminf_{(x,t)\to(\xi,0)} I(x,t) \ge 1$. For this, choose some positive constant $\delta' < \delta$ and $h \in \mathcal{S}(\mathbb{R}^N)$ with $0 \le h \le 1$, $h(\xi) = 1$ and such that $h(y) = 0$ for all y with $|y - \xi| > \delta - \delta'$. Then for each x with $|x - \xi| < \delta'$ the support of h is contained in $\{y \in \mathbb{R}^N : |y - x| \le \delta\}$; therefore

$$\int_{\mathbb{R}^N} h(y)\Gamma_k(x,y,t)w_k(y)dy \;\le\; I(x,t)$$

for all (x,t) with $|x - \xi| < \delta'$. But according to Theorem 4.7 we have

$$\lim_{(x,t)\to(\xi,0)} \int_{\mathbb{R}^N} h(y)\Gamma_k(x,y,t)\, w_k(y)dy \;=\; h(\xi) \;=\; 1.$$

This finishes the proof. □

It is still open whether our solution of the Cauchy problem (4.1) is unique within an appropriate class of functions. As in the classical case, this follows from an maximum principle for the generalized heat operator on $\mathbb{R}^N \times (0,\infty)$. The first step is the following weak maximum principle for H_k on bounded domains. It is proved by a similar method as used in Theorem 4.2. By virtue of Lemma 4.1, this proof is litterally the same as the standard proof in the classical case (see e.g. [J]) and therefore omitted here.

4.12. Proposition. *Suppose that $\Omega \subset \mathbb{R}^N$ is open, bounded and G-invariant. For $T > 0$ set*

$$\Omega_T := \Omega \times (0,\infty) \quad \textit{and} \quad \partial_*\Omega_T := \{(x,t) \in \partial\Omega_T :\, t = 0 \;\textit{or}\; x \in \partial\Omega\,\}.$$

Assume further that $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies $H_k u \ge 0$ in Ω_T. Then

$$\max_{\overline{\Omega}_T}(u) \;=\; \max_{\partial_*\Omega_T}(u)\,.$$

Under a suitable growth condition on the solution, this maximum principle may be extended to the case where $\Omega = \mathbb{R}^N$. The proof is adapted from the one in [dB] for the classical case.

4.13. Theorem. (Weak maximum principle for H_k on $\mathbb{R}^N$.) *Let $S_T := \mathbb{R}^N \times (0,T)$ and suppose that $u \in C^2(S_T) \cap C(\overline{S}_T)$ satisfies*

$$\begin{cases} H_k u \ge 0 & \textit{in } S_T, \\ u(\,.\,,0) = f\,, \end{cases}$$

where $f \in C_b(\mathbb{R}^N)$ is real-valued. Assume further that there exist positive constants C, λ, r such that

$$u(x,t) \;\le\; C\cdot e^{\lambda|x|^2} \quad \textit{for all } (x,t) \in S_T \quad \textit{with } |x| > r.$$

Then $\quad \max_{\overline{S}_T}(u) \;\le\; \sup_{\mathbb{R}^N}(f).$

Proof. Let us first assume that $8\lambda T < 1$. For fixed $\epsilon > 0$ set

$$v(x,t) := u(x,t) - \epsilon \cdot \frac{1}{(2T-t)^{\gamma+N/2}} \exp\left\{\frac{|x|^2}{4(2T-t)}\right\}, \quad (x,t) \in \mathbb{R}^N \times [0,2T).$$

By Lemma 4.3, v satisfies $H_k v = H_k u \geq 0$ in S_T. Now fix some constant $\rho > r$ and consider the bounded cylinder $\Omega_T = \Omega \times (0,T)$ with $\Omega = \{x \in \mathbb{R}^N : |x| < \rho\}$. Setting $M := \sup_{\mathbb{R}^N}(f)$, we have $v(x,0) < u(x,0) \leq M$ for $x \in \overline{\Omega}$. Moreover, for $|x| = \rho$ and $t \in (0,T]$

$$v(x,t) \leq Ce^{\lambda\rho^2} - \epsilon \cdot \frac{1}{(2T)^{\gamma+N/2}}\, e^{\rho^2/8T}.$$

As $\lambda < (8T)^{-1}$, we see that $v(x,t) \leq M$ on $\partial_*\Omega_T$, provided that ρ is large enough. Then by Prop. 4.12 we also have $v(x,t) \leq M$ on $\overline{\Omega}_T$. As $\rho > r$ was arbitrary, it follows that $v(x,t) \leq M$ on $\overline{S}_T$. As $\epsilon > 0$ was arbitrary as well, this implies that $u(x,t) \leq M$ on $\overline{S}_T$. If $8\lambda T \geq 1$, we may subdivide S_T into finitely many adjacent open strips of width less than $1/8\lambda$ and apply the above conclusion repeatedly. □

4.14. Corollary. *The solution of the Cauchy problem (4.1) according to Theorem 4.11 is unique within the class of functions $u \in C^2(S_T) \cap C(\overline{S}_T)$ which satisfy the following exponential growth condition: There exist positive constants C, λ, r such that*

$$|u(x,t)| \leq C \cdot e^{\lambda|x|^2} \quad \textit{for all } (x,t) \in S_T \textit{ with } |x| > r.$$

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